Affine connection

$\Gamma'(\omega)$ s.t. $\quad \Gamma(a\omega) = a\Gamma(\omega)a^{-1} + a\,da^{-1}$

note that $\quad \underbrace{d(aa^{-1})}_{=0} = da\,a^{-1} + a\,da^{-1}$

$$\Rightarrow \quad \Gamma(a\omega) = a\Gamma(\omega)a^{-1} - da\,a^{-1}$$

Prop

At every point $p \in M$ there exists ω s.t. $\Gamma(\omega)_p = 0$.

Proof

If $\Gamma(\omega)_p \neq 0$. then

$$\underbrace{\Gamma(a\omega)_p}_{=0} = a(p)\,\Gamma(\omega)_p\, a^{-1}(p) - (da)_p\, a^{-1}(p)$$

$$\Rightarrow a(p)\Gamma(\omega)_p = (da)_p.$$

Thus it is enough to take $a : U \to GL(n, \mathbb{R})$

s.t. $\quad a(p) = 1$

$\quad\quad (da)_p = \Gamma(\omega)_p.$

$$\Rightarrow \quad \Gamma(a\omega)_p = 0. \quad \text{a.}$$

When we can gauge Γ to zero in a neighbourhood?

$$\Gamma(a\omega) = a\Gamma(\omega)a^{-1} - da\, a^{-1}$$
$$\overset{\parallel}{0}$$

$$\iff \quad da = a\Gamma(\omega).$$

$$\Rightarrow 0 = d^2 a = da \wedge \Gamma(\omega) + a\, d\Gamma(\omega) =$$
$$= a\left(\Gamma(\omega)\wedge\Gamma(\omega) + d\Gamma(\omega)\right)$$

$\Gamma(a\omega) = 0$ only if $\underline{\underline{\Omega(\omega) = d\Gamma(\omega) + \Gamma(\omega)\wedge\Gamma(\omega) = 0}}$

if these equations are satisfied then by Cauchy-Kowalewski

$da = a\Gamma(\omega)$ has a unique solution.

<u>Fact</u>

Check that

$$\Omega^\mu{}_\nu(\omega) = d\Gamma^\mu{}_\nu(\omega) + \Gamma^\mu{}_\rho(\omega)\wedge\Gamma^\rho{}_\nu(\omega)$$

is a 2-form of type Ad

$$\Omega^\mu{}_\nu(a\omega) = a^\mu{}_\alpha\, \Omega^\alpha{}_\beta(\omega)\, a^{-1\beta}{}_\nu \ .$$

<u>Curvature 2-form</u>!

Another connonical form:

$$\theta^\mu(\omega) := \omega^\mu \qquad \text{— connonical 1-form of type id.}$$

$$D\theta^\mu = d\theta^\mu + \Gamma^\mu{}_\nu \wedge \theta^\nu = \Theta^\mu$$

torsion 2-form.

$$\Theta^\mu(\omega) = d\omega^\mu + \Gamma^\mu{}_\nu(\omega) \wedge \omega^\nu.$$

If $d\omega^\mu|_p = 0$ and $\Gamma^\mu{}_\nu(\omega)_p = 0 \Rightarrow \Theta^\mu(\omega)_p = 0$

$$\Theta^\mu(\omega)_p = 0$$
$$\Downarrow$$
$$\omega' \text{ s.t. } \Gamma^\mu{}_\nu(\omega')_p = 0$$
$$\Downarrow$$
$$d\omega'^\mu|_p = 0$$

$$\boxed{\Theta(\omega)_p = 0 \Leftrightarrow \exists \omega \text{ s.t. } d\omega^\mu|_p = 0 \text{ and } \Gamma^\mu{}_\nu(\omega)_p = 0}$$

<u>Ricci formula</u>

$$\Gamma(a\omega) = a\Gamma(\omega)a^{-1} - da\,a^{-1}$$

α – k-form of type ρ. $\Rightarrow D\alpha = d\alpha + \rho'(\Gamma)\wedge\alpha$;

in addition at every point $p \in M$ we can find ω s.t. $\Gamma(\omega)_p = 0$. This is very useful during calculations!

Fact

Let β be an ℓ-form of type σ on M (e.g. $\beta = D\alpha$)

$$\left(\Gamma(\omega)_p = 0 \text{ and } \beta(\omega)_p = 0\right) \Rightarrow \beta(a\omega)_p = 0$$

(because $\beta(a\omega) = \sigma(a)\beta(\omega)$).

Example:

$$\begin{matrix} \alpha_1 \\ k_1, \rho_1 \end{matrix} \; ; \; \begin{matrix} \alpha_2 \\ k_2, \rho_2 \end{matrix} \Rightarrow \begin{matrix} \alpha_1 \wedge \alpha_2 \\ k_1 + k_2, \; \rho_1 \otimes \rho_2 \end{matrix}$$

$\Downarrow$

$$D(\alpha_1 \wedge \alpha_2) = D\alpha_1 \wedge \alpha_2 + (-1)^{k_1} \alpha_1 \wedge D\alpha_2$$

because in a frame in which $\Gamma(\omega)_p = 0$ we have

$$\left[D(\alpha_1 \wedge \alpha_2) - \left(D\alpha_1 \wedge \alpha_2 + (-1)^{k_1}\alpha_1 \wedge D\alpha_2\right)\right]_p \overset{\Gamma(\omega)_p = 0}{=} d(\alpha_1 \wedge \alpha_2)_p + - \left(d\alpha_1 \wedge \alpha_2 + (-1)^{k_1} \alpha_1 \wedge d\alpha_2\right)_p = 0.$$

Better example: Ricci formula

$$\boxed{D^2\alpha = \rho'(\Omega) \wedge \alpha}$$

$$\begin{aligned} D\left[d\alpha + \rho'(\Gamma) \wedge \alpha\right] &= \rho'(d\Gamma) \wedge \alpha + \text{terms linear in } \Gamma \\ &= \rho'(\Omega) \wedge \alpha + \text{terms linear in } \Gamma \\ &\underset{\Gamma(\omega)_p = 0}{=} \rho'(\Omega) \wedge \alpha \quad \text{in this frame;} \end{aligned}$$

hence, since

$D^2\alpha - \rho'(\Omega)\wedge\alpha$ is $k+2$ form of type ρ

and $(D^2\alpha - \rho'(\Omega)\wedge\alpha)_p = 0$ in this frame
$\Rightarrow$ in every frame!

Bianchi identities

(II): $D\Omega^\mu{}_\nu = d\Omega^\mu{}_\nu + \Gamma^\mu{}_\rho \wedge \Omega^\rho{}_\nu - \Gamma^\rho{}_\nu \wedge \Omega^\mu{}_\rho =$

$= d\left(d\Gamma^\mu{}_\nu + \Gamma^\mu{}_\rho \wedge \Gamma^\rho{}_\nu\right) +$ terms linear in Γ $=$

$= d^2\Gamma^\mu{}_\nu +$ terms linear in $\Gamma = 0 +$ terms linear in Γ

$$\underset{\Gamma(\omega)_p = 0}{\uparrow}{=}\, 0 \quad \Rightarrow \quad \boxed{D\Omega^\mu{}_\nu = 0} \qquad \text{II}^{nd} \text{ B.I.}$$

(I): $D\Theta^\mu = D^2\theta^\mu \underset{\text{Ricci formula}}{\overset{}{=}} \Omega^\mu{}_\nu \wedge \theta^\nu$

$$\boxed{D\Theta^\mu = \Omega^\mu{}_\nu \wedge \theta^\nu} \qquad \text{I}^{st} \text{ B.I.}$$

If α is a 0-form of type ρ:

$$D\alpha^A(\omega) = \omega^\mu \nabla_{X_\mu} \alpha^A$$

in other words:

$$\nabla_{X_\mu} \alpha^A = X_\mu \lrcorner\, D\alpha^A(\omega)$$

$$\boxed{\nabla_X \alpha^A = X \lrcorner\, D\alpha^A(\omega)}$$

Exercise: calculate

$$\nabla_{[\mu} \nabla_{\nu]} \alpha^A = ?$$

How torsion and curvature look in terms of ∇.?

$$\Theta^\mu = d\theta^\mu + \Gamma^\mu{}_\nu \wedge \theta^\nu$$

; by Maurer-Cartan:

$$d\theta^\mu = -\tfrac{1}{2} C^\mu{}_{\nu\rho}\, \theta^\nu \wedge \theta^\rho$$

$$\boxed{X_\alpha \lrcorner X_\beta \lrcorner \Theta^\mu} = X_\alpha \lrcorner X_\beta \lrcorner \left(-\tfrac{1}{2} C^\mu{}_{\nu\rho}\, \theta^\nu \wedge \theta^\rho + \Gamma^\mu{}_\nu \wedge \theta^\nu\right) =$$

$$= \boxed{-C^\mu{}_{\beta\alpha} + \Gamma^\mu{}_{\alpha\beta} - \Gamma^\mu{}_{\beta\alpha}} \quad (1)$$

$X_\alpha \lrcorner \theta^\mu = \delta_\alpha{}^\mu$

$\Uparrow$

Observe that

$$\boxed{\begin{aligned} &X_\alpha \lrcorner \theta^\mu(\omega) = \\ &= X_\alpha \lrcorner \omega^\mu = \delta_\alpha{}^\mu \end{aligned}}$$

introduce

$\delta_\alpha{}^\mu(\omega) = \delta_\alpha{}^\mu$

$\uparrow$

Scalar 0-form

Define

$T : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ by:

$$\boxed{T(Y,Z) = \nabla_Y Z - \nabla_Z Y - [Y,Z].}$$

Properties

1) $T(Y,Z) = -\ \ (Z,Y)$ ✓

2) T is f-linear: e.g.

$$T(fY,Z) = f\nabla_Y Z - \nabla_Z(fY) - [fY,Z] =$$

$$= f\nabla_Y Z - \cancel{Z(f)Y} - f\nabla_Z Y - f[Y,Z] + \cancel{Z(f)Y}$$ ✓

$$T(Y,Z) = T^\mu(Y,Z)\, X_\mu$$

$$\boxed{X_\alpha \lrcorner X_\beta \lrcorner T^\mu} = T^\mu(X_\beta, X_\alpha) = \left(\nabla_{X_\beta} X_\alpha - \nabla_{X_\alpha} X_\beta - [X_\beta, X_\alpha]\right)^\mu =$$

$$= \boxed{\Gamma^\mu{}_{\alpha\beta} - \Gamma^\mu{}_{\beta\alpha} - C^\mu{}_{\beta\alpha}} \quad (2)$$

$$\Longrightarrow T^\mu = \Theta^\mu$$

or $\boxed{T = \Theta^\mu X_\mu}$

In a similar way:

$$\Omega^\mu{}_\nu = d\Gamma^\mu{}_\nu + \Gamma^\mu{}_\rho \wedge \Gamma^\rho{}_\nu \quad \text{- curvature}$$

Define:

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

$$\boxed{R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z}$$

Properties

1) $R(X,Y) = -R(Y,X)$
2) R is f-linear in each argument

$$R(X,Y)Z = R^\alpha{}_\beta (X,Y)\, \theta^\beta(Z) X_\alpha$$

and $\mathrm{End}(\mathbb{R}^n)$-valued 2-forms $R^\alpha{}_\beta$ coincide with $\Omega^\alpha{}_\beta$

$$R^\alpha{}_\beta = \Omega^\alpha{}_\beta$$

$$\Longrightarrow \quad \boxed{R = \Omega^\alpha{}_\beta\, \theta^\beta \otimes X_\alpha}$$

or $\boxed{R(\cdot,\cdot) X_\mu = \Omega^\alpha{}_\mu X_\alpha}$